Dealing with Range Anxiety in Mean Estimation via Statistical Queries

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Abstract
In the statistical query (SQ) model [Kea98, FGR+12] an algorithm has access to an SQ oracle for the input distribution \( D \) over \( X \) instead of i.i.d. samples from \( D \). Given a query function \( \phi : X \to [-1, 1] \), the oracle returns an estimate of \( \mathbb{E}_{x \sim D}[\phi(x)] \) within some tolerance \( \tau \). In a variety of natural problems it is necessary to estimate expectations of functions whose standard deviation is much smaller than the range. In this note we describe a nearly optimal algorithm for estimation of such expectations via statistical queries. As applications, we give algorithms for high dimensional mean estimation in the SQ model and in the distributed setting where only a single bit is communicated from each sample.

1 Overview
We consider the problem of estimating the expectation \( D[\phi] = \mathbb{E}_{x \sim D}[\phi(x)] \), where \( D \) is the unknown input distribution over \( X \) and \( \phi : X \to \mathbb{R} \). We study this problem in the statistical query (SQ) model in which an algorithm has access to a statistical query oracle for \( D \) in place of i.i.d. samples from \( D \) as in the traditional setting of statistics and machine learning. The most commonly studied SQ oracle was introduced by Kearns [Kea98] and gives an estimate of the mean of any bounded function with fixed tolerance.

**Definition 1.1.** Let \( D \) be a distribution over a domain \( X \) and \( \tau > 0 \). A statistical query oracle \( \text{STAT}_D(\tau) \) is an oracle that given as input any function \( \phi : X \to [-1, 1] \), returns some value \( v \) such that \( |v - D[\phi]| \leq \tau \).

A special case of statistical queries are counting or linear queries in which the distribution \( D \) is uniform over the elements of a given database \( S \in X^n \). This setting is studied extensively in the literature on differential privacy (see [DR14] for an overview) and our discussion applies to this setting as well.

Tolerance \( \tau \) of statistical queries roughly corresponds to the number of random samples in the traditional setting. Namely, the Chernoff-Hoeffding bounds imply that \( n \) i.i.d. samples allow estimation of \( D[\phi] \) with tolerance \( \tau = \Theta(1/\sqrt{n}) \) (with high probability). However, if the variance of \( \phi(x) \) is low then \( n \) random samples will likely give a more accurate estimate. To address this discrepancy a somewhat stronger oracle was introduced in [FGR+12].

**Definition 1.2.** Let \( D \) be a distribution over a domain \( X \) and \( n > 0 \). A statistical query oracle \( \text{VSTAT}_D(n) \) is an oracle that given as input any function \( \phi : X \to [0, 1] \) returns a value \( v \) such that \( |v - p| \leq \max \left\{ \frac{1}{n}, \sqrt{\frac{p(1-p)}{n}} \right\} \), where \( p = D[\phi] \).

Clearly \( \text{VSTAT}_D(n) \) is at least as strong as \( \text{STAT}_D(1/\sqrt{n}) \) (but no stronger than \( \text{STAT}_D(1/n) \)). When the range of \( \phi \) is \( \{0, 1\} \), one way to think about \( \text{VSTAT} \) is as providing a confidence interval for the bias.
which well as possible using VSTAT. There exists a statistical query algorithm that given an integer $n$, $\phi : X \to \mathbb{R}$ such that $D[\phi^2] \leq B^2$, outputs a value $\tilde{v}$ such that $|D[\phi] - \tilde{v}| \leq \sigma^2 (\log(n+3) + \zeta$, where $\sigma^2 \approx D[\phi^2] - D[\phi]^2$. The algorithm uses $O(\log(Bn/\zeta))$ queries to VSTAT($n$).

Theorem 1.3. There exists a statistical query algorithm that given an integer $n$, $\zeta > 0$, $B > 0$ and $\phi : X \to \mathbb{R}$ such that $D[\phi^2] \leq B^2$, outputs a value $\tilde{v}$ such that $|D[\phi] - \tilde{v}| \leq \sigma \sqrt{n+3} + \zeta$, where $\sigma \approx D[\phi^2] - D[\phi]^2$. The algorithm uses $O(\log(Bn/\zeta))$ queries to VSTAT($n$).

The algorithm works by estimating various percentiles to restrict the range of $\phi$ and get a rough estimate of its expectation (via Lemma 2.1). It then splits the restricted range into a logarithmic number of intervals in which $\phi - \tilde{v}$ has small dynamic range, where $\tilde{v}$ is the estimate of $D[\phi]$ within $O(\sigma)$. We also give a simpler algorithm and bounds for the case when we have an upper bound on $\sigma$ and/or already have an estimate of $D[\phi]$ within $O(\sigma)$ (see Lemma 3.2). In this case the algorithm’s queries are non-adaptive, that is, query functions do not depend on answers to previous queries.
Applications: As an example application we consider the problems of mean vector estimation and stochastic optimization in the SQ model [FGV15]. In the $\ell_2$ mean vector estimation problem the goal is to estimate $\bar{x} \doteq E_{x \sim D}[x]$ within $\epsilon$ in the $\ell_2$ norm, where $D$ is a distribution over vectors in $\mathbb{R}^d$. This problem is central to implementation of gradient-based (or first-order) optimization methods in the stochastic setting. In [FGV15] it was showed that using either a randomly rotated basis or Kashin’s representation [LV10] this problem can be solved using $O(d)$ queries to $\text{STAT}_D(\Omega(\epsilon))$ when $D$ is supported on the unit ball. Our results give a simpler and more general algorithm for the problem using the $\text{VSTAT}_D(\tilde{O}(1/\epsilon^2))$ oracle.

Corollary 1.4. There exists a statistical query algorithm that given $\epsilon \in (0,1)$ and $B > \epsilon$, for any distribution $D$ over $\mathbb{R}^d$ such that $E_{x \sim D}[||x||^2_2] \leq B^2$ and $E_{x \sim D}[||x - \bar{x}||^2_2] \leq 1$ outputs a vector $\hat{x}$ such that $||\hat{x} - \bar{x}||_2 \leq \epsilon$, where $\bar{x} \doteq E_{x \sim D}[x]$. The algorithm uses $O(d(\log(dB/\epsilon)))$ queries to $\text{VSTAT}_D(\Omega((\log(1/\epsilon)/\epsilon)^2))$.

Corollary 1.4 implies that the optimization algorithms in [FGV15] can be extended from the setting in which the (sub-)gradients of all the functions in the support of the input distribution are uniformly upper-bounded to the setting in which only variance of (sub-)gradient vectors is upper-bounded. These corollaries can in turn be translated to the 1-bit sampling model and we describe an example corollary in Section 4.

2 Preliminaries

For integer $n \geq 1$ let $[n] \doteq \{1, \ldots, n\}$. Random variables are denoted by bold letters, e.g., $x$. We denote the indicator function of an event $A$ (i.e., the function taking value zero outside of $A$, and one on $A$) by $1_{\{A\}}$.

We will use the following notation for a truncation operation. For a real value $z$ and $a \in \mathbb{R}^+$, let

$$m_a(z) := \begin{cases} 
 z & \text{if } |z| \leq a \\
 a & \text{if } z > a \\
 -a & \text{if } z < -a.
\end{cases}$$

Let $r_a(z) = z - m_a(x)$.

We will need the following lemma about estimation of distribution percentiles using the VSTAT oracle.

Lemma 2.1. Let $\phi$ be a function with a finite range $Z \subset \mathbb{R}$. There exists an SQ algorithm that given $p$ and $\delta$ such that $1 \geq p \geq 2\delta > 0$, outputs a point $a \in Z$ such that $\Pr_D[\phi \geq a] \geq p - \delta$ and $\Pr_D[\phi > a] < p$. The algorithm uses $\log(|Z|)$ queries to $\text{VSTAT}_D(4p/\delta^2)$.

Proof. For any $z \in Z$, let $p_z \doteq \Pr_D[\phi \leq z]$. We perform a binary search to find the largest point $\hat{z}$ such that the estimate of $\Pr_D[\phi \geq \hat{z}]$ given by $\text{VSTAT}_D(4p/\delta^2)$ is at least $p - \delta/2$. We denote the point by $a$ and refer to estimates we have obtained by $\tilde{p}_a$. By definition, $\tilde{p}_a \geq p - \delta/2$ and therefore $p_a \geq p - \delta$, since otherwise

$$\tilde{p}_a < p - \delta + \max \left\{ \sqrt{\frac{p - \delta}{4p/\delta^2}}, \frac{\delta^2}{4p} \right\} < p - \delta/2 - \delta/2 = p - \delta/2.$$

On the other hand for the smallest point in $Z$ larger than $a$ (denote it by $a_+\text{)}$ we know that $\tilde{p}_{a_+} < p - \delta/2$. This implies that $p_{a_+} \leq p$ since otherwise

$$\tilde{p}_{a_+} > p_{a_+} - \max \left\{ \sqrt{\frac{p_{a_+}}{4p/\delta^2}}, \frac{\delta^2}{4p} \right\} = p_{a_+} - \frac{\delta}{2} \sqrt{\frac{p_{a_+}}{p}} \geq \sqrt{p \cdot p_{a_+} - \delta/2} > p - \delta/2.$$

This means that $\Pr_D[\phi > a] < p$.  \qed
As a special case we obtain the following corollary.

**Corollary 2.2.** Let \( \phi \) be a function over \( X \) with a finite range \( Z \subset \mathbb{R} \). There exists an SQ algorithm that given an integer \( n \) outputs a point \( a \in Z \) such that \( \Pr_D[\phi \geq a] \geq 8/n \) and \( \Pr_D[\phi \leq a] \geq 16/n \). The algorithm uses \( \log(|Z|) \) queries to \( VSTAT_D(n) \).

One natural way to apply this lemma in the continuous setting is to first discretize the range with some step size \( \zeta \) and then use Lemma 2.1. This leads to the following version of Lemma 2.1.

**Corollary 2.3.** Let \( \phi : X \to [-B, B] \) be a function and \( \zeta > 0 \). There exists an SQ algorithm that given \( p \) and \( \delta \) such that \( 1 \geq p \geq 2\delta > 0 \), outputs a point \( a \) that is an integer multiple of \( \zeta \) such that \( \Pr_D[\phi \geq a] \geq p - \delta \) and \( \Pr_D[\phi \geq a + \zeta] < p \). The algorithm uses \( \lceil \log(B/\zeta) \rceil \) queries to \( VSTAT_D(4p/\delta^2) \).

3 Mean estimation for random variables with bounded variance

We start by showing that the dependence on \( \sqrt{D[\phi]} \) in the accuracy of VSTAT can be strengthened to \( \sqrt{D[\phi^2]} \) at the expense of logarithmic factors in the complexity.

**Lemma 3.1.** There exists a statistical query algorithm that, for \( n, R > 0 \), any function \( \phi : X \to [0, R] \) and any input distribution \( D \) over \( X \), outputs a value \( v \) such that \( |D[\phi] - v| \leq 3R \sqrt{n} + \frac{s \log n}{\sqrt{n}} \), where \( s = \sqrt{D[\phi^2]} \). The algorithm uses at most \( \log n \) (non-adaptive) queries to \( VSTAT_D(n) \).

**Proof.** We assume for simplicity that \( R = 1 \) since we can always scale \( \phi \) to this setting and then scale back the result. We let \( t = \lceil \log n \rceil \) and observe that

\[
D[\phi] = E_D[\phi(x) \cdot 1_{\{\phi(x) \in [0, 2^{-t}]\}}] + \sum_{i \in [t]} E_D[\phi(x) \cdot 1_{\{\phi(x) \in [2^{-i}, 2^{-i+1}]\}}]. \tag{1}
\]

For every \( i \in [t] \), we define \( \phi_i \) to be the restriction of \( \phi \) to values in the interval \( (2^{-i}, 2^{-i+1}] \), scaled and shifted to the range \( [0, 1] \). Namely

\[
\phi_i(x) = 2^i \cdot \phi(x) \cdot 1_{\{\phi(x) \in [2^{-i}, 2^{-i+1}]\}} - 1.
\]

Using this definition, we can rewrite eq. (1) as

\[
D[\phi] = E_D[\phi(x) \cdot 1_{\{\phi(x) \in [0, 2^{-t}]\}}] + \sum_{i \in [t]} 2^{-i} \cdot (D[\phi_i] + 1). \tag{2}
\]

For every \( i \in [t] \), we query function \( \phi_i \) to \( VSTAT_D(n) \) to get an estimate \( v_i \) of \( D[\phi_i] \). By Chebyshev’s inequality,

\[
D[\phi_i] \leq \Pr_D[\phi_i(x) > 2^{-i}] \leq \frac{D[\phi^2]}{2^{-2i}} = 2^{2i} \cdot s^2.
\]

Therefore, by the definition of \( VSTAT_D(n) \),

\[
|v - D[\phi_i]| \leq \max \left\{ \frac{1}{n}, \sqrt{\frac{D[\phi_i]}{n}} \right\} \leq \frac{1}{n} + \sqrt{\frac{D[\phi_i]}{n}} \leq \frac{1}{n} + \frac{2^i s}{\sqrt{n}}. \tag{3}
\]

Let \( v = \sum_{i \in [t]} 2^{-i} \cdot (v_i + 1) \).

Then, by eq. (2) and eq. (3) we get that
where we used Chebyshev’s inequality to obtain a bound on $\epsilon \frac{\zeta}{Bn}$ the closest multiple of $\log(4Bn/\zeta)$ such that $D \geq 2$. Lemma 3.3. There exists a statistical query algorithm that given an integer $n$, $\zeta > 0$, $B > 0$, and $\phi : X \to \mathbb{R}^+$ such that $D[\phi^2] \leq B^2$, outputs a value $v$ such that $|D[\phi] - v| \leq \epsilon$. The algorithm uses at most $3 \log(B/\epsilon)$ (non-adaptive) queries to $\text{VSTAT}_D((4B \log(B/\epsilon)/\epsilon)^2)$.

Proof. We first assume that $B = 1$. Observe that we can truncate the range of $\phi$ to $[0, a]$ for $a = 4/\epsilon$ without a significant change in the expectation. Namely, we claim that $|D[m_a(\phi)] - D[\phi]| \leq \epsilon/4$. To see this when $a = 4/\epsilon$ note that $\phi - m_a(\phi) = r_a(\phi)$ and

$$E[r_a(\phi)] = \int_0^\infty \Pr[r_a(\phi) \geq t] dt = \int_0^\infty \Pr[\phi \geq t + a] dt = \int_a^\infty \Pr[\phi \geq t] dt \leq \int_a^\infty \frac{1}{t^2} dt = \frac{1}{a} \leq \epsilon/4,$$

where we used Chebychev’s inequality to obtain a bound on $\Pr[\phi \geq t]$.

Now to estimate $D[m_a(\phi)]$ we use Lemma 3.1 with $n = (4 \log(1/\epsilon)/\epsilon)^2$. For the given range and our assumption that $D[\phi^2] \leq 1$, this leads to an error of at most

$$12 \epsilon \left(\frac{4 \log(1/\epsilon)/\epsilon}{2 \log(1/\epsilon)/\epsilon}\right)^2 + \frac{2 \log(4 \log(1/\epsilon)/\epsilon)}{4 \log(1/\epsilon)/\epsilon} \leq \frac{3 \epsilon}{4 \log(1/\epsilon)^2} + \frac{\epsilon \log(4 \log(1/\epsilon)/\epsilon)}{2 \log(1/\epsilon)} \leq \frac{3 \epsilon}{4},$$

where we assume that $\epsilon \leq 1/16$ to obtain the last inequality. Altogether this implies that the output of the Lemma on the truncated function will have an error of at most $\epsilon$. Finally, to generalize the analysis to any $B > 0$ we simply scale the random variable by $1/B$ and estimate its mean within $\epsilon/B$.

Next, using Lemma 2.1 we describe a procedure that can estimate the mean without knowing a upper-bound on the second moment.

Lemma 3.3. There exists a statistical query algorithm that given an integer $n$, $\zeta > 0$, $B > 0$ and $\phi : X \to \mathbb{R}^+$ such that $D[\phi^2] \leq B^2$, outputs a value $v$ such that $|D[\phi] - v| \leq \frac{s(\log(n + 6)/\sqrt{n})}{\zeta} + \zeta$, where $s = \sqrt{D[\phi^2]}$. The algorithm uses $\log(4Bn/\zeta^2)$ queries to $\text{VSTAT}_D(n)$.

Proof. We start by truncating and discretizing the range of $\phi$, namely let $\psi$ be equal to $\phi$ rounded down to the closest multiple of $\zeta/2$ and truncated at $2B/\zeta$. As in Lemma 3.2, we note that the condition $D[\phi^2] \leq B^2$
implies that truncation to range \([0, 2B/\zeta]\) step can affect the expectation by at most \(\zeta/2\). Clearly, rounding down to the closest multiple of \(\zeta/2\) also affects the expectation by at most \(\zeta/2\). This means that \(\psi\) has range \(Z\) of size at most \(4B/\zeta^2\), \(|D[\phi] - D[\psi]| \leq \zeta\) and \(D[\psi^2] \leq s^2\).

We now further truncate the range of \(\psi\) to exclude values that are in the top \(8/n\) percentile. Namely, by Corollary 2.3, we can find a value \(a \in Z\) such that \(\Pr_D[\psi \geq a] \geq 8/n\) and \(\Pr_D[\psi > a] < 16/n\). Let \(\psi_a(x)\) be defined as \(m_a(\psi(x))\).

We first observe that \(s^2 = D[\psi^2] \geq a^2 \cdot 8/n\). Therefore \(a \leq \frac{s \sqrt{n}}{\sqrt{8}}\). Next note that

\[
|D[\psi] - D[\psi_a]| \leq \sum_{z \in Z, z \geq a} z \cdot \Pr_D[\psi = z]
\]

\[
\leq \sqrt{\sum_{z \in Z, z \geq a} \Pr_D[\psi = z]} \cdot \sqrt{\sum_{z \in Z, z > a} z^2 \cdot \Pr_D[\psi = z]}
\]

\[
\leq \sqrt{\Pr_D[\psi > a]} \cdot \sqrt{D[\psi^2]} \leq \frac{4s}{\sqrt{n}},
\]

(4)

where we used the Cauchy-Schwartz inequality to obtain the second line.

We can now apply Lemma 3.4 to \(\psi_a\) and obtain a value \(v\) such that

\[
|D[\psi] - v| \leq \frac{s \log n}{\sqrt{n}} + \frac{3a}{n} = \frac{s \log n + 3s/\sqrt{8}}{\sqrt{n}}.
\]

Combining this with eq. (4) and the properties of \(\psi\) we get the claim.

We can easily extend these results to variables with possibly negative range by estimating the mean in the positive and the negative range separately. Further, one does not necessarily need to split the range at \(0\). Any value \(a\) can be used and the resulting bound will be in terms of \(s_a = \sqrt{D[(\phi - a)^2]}\) instead of \(s_0 = \sqrt{D[\phi^2]}\). Naturally, in order to reduce the error we should use \(a = D[\phi]\) which would give an error in terms of variance \(\sigma^2 = D[(\phi - a)^2] - D[\phi]^2\). The true mean is not known to but, as we show below, we can use an (approximate) median instead.

**Lemma 3.4.** Let \(x\) be a random variable over \(\mathbb{R}\) and let \(a\) be any point such that \(\Pr[x \geq a] \geq 1/3\) and \(\Pr[x \leq a] \geq 1/3\). Then \(\mathbb{E}[(x - a)^2] \leq 4(\mathbb{E}[x^2] - \mathbb{E}[x]^2)\).

**Proof.** Let \(\bar{z} = \mathbb{E}[z]\) and \(\sigma^2 = \mathbb{E}[(z - \bar{z})^2]\). Observe that

\[
\sigma^2 = \mathbb{E}[(z - \bar{z})^2] \geq (\bar{z} - a)^2 \cdot \Pr[z \geq |(\bar{z} - a)|] \geq (\bar{z} - a)^2/3.
\]

Now

\[
\mathbb{E}[(z - a)^2] \leq \sigma^2 + (\bar{z} - a)^2 \leq 4\sigma^2.
\]

Note that for a function with range \(Z\) such an approximate median can be found using \(\log(|Z|)\) queries to \(\text{VSTAT}_D(6)\). Therefore we immediately get the following results:

**Theorem 3.5.** There exists a statistical query algorithm that given an integer \(n\), \(\zeta > 0\), \(B > 0\) and \(\phi : X \rightarrow \mathbb{R}\) such that \(D[\phi^2] \leq B^2\), outputs a value \(v\) such that \(|D[\phi] - v| \leq \frac{\sigma(4 \log n + 3)}{\sqrt{n}} + \zeta\), where \(\sigma^2 = D[\phi^2] - D[\phi]^2\). The algorithm uses \(3 \log(4nB/\zeta^2)\) queries to \(\text{VSTAT}_D(n)\).
An alternative way to state essentially the same result is as follows.

**Corollary 3.6.** There exists a statistical query algorithm that given \( B > \zeta > 0, \epsilon \in (0,1) \), for any distribution \( D \) and function \( \phi : X \to \mathbb{R} \) such that \( D[\phi^2] \leq B^2 \) outputs a value \( v \) such that \( |D[\phi] - v| \leq \sigma + \zeta \). The algorithm uses \( O((\log(B/(\epsilon \zeta))) \) queries to \( VSTAT_D(O((\log(1/\epsilon)/\epsilon)^2)) \), where \( \sigma^2 \doteq D[\phi^2] - D[\phi]^2 \).

## 4 Applications

As application we consider the problems of mean vector estimation. \([\text{FGV15}]\). In the \( \ell_2 \) mean vector estimation problem the goal is to estimate \( \bar{x} = \mathbb{E}_{x \sim D}[x] \) within \( \epsilon \) in \( \ell_2 \) norm, where \( D \) is a distribution over vectors in \( \mathbb{R}^d \). Our results give a simple and general algorithm for solving this problem using the VSTAT oracle.

**Corollary 4.1.** There exists a statistical query algorithm that given \( \epsilon \in (0,1) \) and \( B > \epsilon \), for any distribution \( D \) over \( \mathbb{R}^d \) such that \( \mathbb{E}_{x \sim D}[\|x\|^2] \leq B^2 \) and \( \mathbb{E}_{x \sim D}[\|x - \bar{x}\|^2] \leq 1 \) outputs a vector \( \hat{x} \) such that \( \|\hat{x} - \bar{x}\|_2 \leq \epsilon \), where \( \bar{x} = \mathbb{E}_{x \sim D}[x] \). The algorithm uses \( O(d((\log(dB/\epsilon))) \) queries to \( VSTAT_D(O((\log(1/\epsilon)/\epsilon)^2)) \).

**Proof.** We apply the algorithm given in Thm. \(1.3\) to each of the coordinates of \( x \) with \( \zeta = \epsilon/\sqrt{2d} \). Namely, for each \( i \in [d] \) we use Thm. \(1.3\) to estimate the expectation of function \( \phi_i(x) = x_i \) and let \( \hat{x}_i \) be the result. For every \( i \in [d], \mathbb{E}_{X \sim D}[\|x_i\|^2] \leq B^2 \) and hence, by setting \( n = c(\log(1/\epsilon)/\epsilon)^2 \) for an appropriately chosen constant \( c \), we will obtain that the error in the estimation of coordinate \( i \) is at most

\[
\frac{\epsilon}{\sqrt{2}} \cdot \sqrt{\mathbb{E}_{x \sim D}[\|x_i\|^2] - \bar{x}_i^2} + \frac{\epsilon}{\sqrt{2d}}.
\]

By observing that

\[
\sum_{i \in [d]} \mathbb{E}_{X \sim D}[\|x_i\|^2] - \bar{x}_i^2 = \mathbb{E}_{X \sim D}[\|x - \bar{x}\|^2] \leq 1
\]

we obtain the claim. \( \square \)

These corollaries can in turn be translated to the 1-bit sampling model. For this purpose we define the model formally and state the simulation result of Feldman et al. \([\text{FGR12}]\).

**Definition 4.2 (1-STAT oracle \([\text{BD98}]\)).** Let \( D \) be a distribution over the domain \( X \). The 1-STAT\(_D\) oracle is the oracle that given any function \( h : X \to \{0,1\} \), takes an independent random sample \( x \) from \( D \) and returns \( h(x) \).

**Theorem 4.3 (\([\text{FGR12}]\)).** Let \( n, q > 0 \) be any integers and \( \delta > 0 \). For any algorithm \( A \) that asks at most \( q \) queries to \( VSTAT_D(n) \) there exists an algorithm \( A' \) that, with probability at least \( 1 - \delta \), provides valid for \( VSTAT(n) \) answers to all the queries of \( A \). \( A' \) uses \( O(qn \cdot \log(q/\delta)) \) queries to 1-STAT\(_D\).

We remark that if the SQ algorithm \( A \) is non-adaptive then the simulation in Theorem \(4.3\) produces a non-adaptive algorithm. This is particularly useful in the distributed setting since it allows the bits from each of the samples to be communicated in parallel.

Combining Corollary \(3.6\) and Theorem \(4.3\) we obtain the following algorithm for estimating the mean.

**Corollary 4.4.** There exists an algorithm that given \( \epsilon, \delta \in (0,1) \) and \( B > \epsilon \), for any distribution \( D \) over \( \mathbb{R}^d \) such that \( \mathbb{E}_{X \sim D}[\|x\|^2] \leq B^2 \) and \( \mathbb{E}_{X \sim D}[\|x - \bar{x}\|^2] \leq 1 \) outputs a vector \( \hat{x} \) such that, with probability at least \( 1 - \delta \), \( \|\hat{x} - \bar{x}\|_2 \leq \epsilon \), where \( \bar{x} = \mathbb{E}_{X \sim D}[x] \). The algorithm uses \( O(d/\epsilon^2 \cdot \log(B/(\delta \epsilon))) \) queries to 1-STAT\(_D\).
References


