ON RECURRENCE AND TRANSIENCE IN HEAVY-TAILED GSMP’S

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Discrete-Event Stochastic Systems

- System changes **state** when **events** occur
  - Stochastic state changes
  - At strictly increasing sequence of random times
- Underlying stochastic process \( \{ X(t): t \geq 0 \} \)
  - \( X(t) = \) state of system at time \( t \) (a random variable)
  - Piecewise-constant sample paths
  - Typically **not** a continuous-time Markov chain
Generalized Semi-Markov Processes [Matthes62, Whitt80]

- Classical model for discrete-event stochastic systems
  - Subsumes: queueing networks, SMPs, CTMCs, SPNs
  - Central to simulation theory

- Building blocks
  - $S =$ set of states (finite or countably infinite)
  - $E =$ set of events (finite)
  - $E(s) =$ active events in state $s$
  - $p(s'; s, e^*) =$ state-transition probability

- One clock per event: records remaining time until occurrence
Clocks (Event Scheduling)

- Active events compete to trigger state transition
  - The clock that runs down to 0 first is the “winner”
  - Can have simultaneous event occurrence: $p(s'; s, E^*)$

- At a state transition $s \xrightarrow{e^*} s'$: three kinds of events
  - New events: Clock for $e'$ set according to $F(x; e')$
  - Old events: Clocks continue to run down
  - Cancelled events: Clock readings are discarded
The GSMP Process

- The continuous-time process: \( \{ X(t) : t \geq 0 \} \)
  - \( X(t) = \) the state at time \( t \)
  - A very complicated process

- Defined in terms of Markov chain \( \{ (S_n, C_n) : n \geq 0 \} \)
  - System observed after the \( n \)th state transition
  - \( S_n = \) the state
  - \( C_n = (C_{n,1}, \ldots, C_{n,M}) = \) the clock-reading vector
  - Chain defined via GSMP building blocks
  - initial distribution \( \mu \) on state and clocks
Definition of the GSMP

\[ X(t) = S_{N(t)} \]
Example: GI/G/1 Queue

- $X(t) = \text{"number of jobs in system at time } t\text{"}$
- $S = \{0, 1, 2, \ldots\}$
- $E = \{e_1, e_2\}$
  - $e_1 = \text{"arrival"}$
  - $e_2 = \text{"service completion"}$
- $E(0) = \{e_1\}$ and $E(s) = \{e_1, e_2\}$ for $s > 0$
- $p(s + 1; s, e_1) = 1$ and $p(s - 1; s, e_2) = 1$
- $F(\cdot; e_1) = F_{\text{interarrival}}$ and $F(\cdot; e_2) = F_{\text{service}}$
Harris Recurrence: A Basic Form of Stability

Definition for general chain \( \{ Z_n : n \geq 0 \} \) with state space \( \Gamma \)

\[
P_z \{ Z_n \in A \text{ i.o.} \} = 1, \ z \in \Gamma \quad \text{whenever} \quad \phi(A) > 0
\]

- \( \phi \) is a recurrence measure (often “Lebesgue-like”)
- Every “dense enough” set is hit infinitely often w.p. 1
- No “wandering off to \( \infty \)”
- Chain admits invariant measure \( \pi_0: \int P(z, A) \pi_0(dz) = \pi_0(A) \)

Positive Harris recurrence:

- Chain admits invariant probability measure \( \pi \)
- \( P_\pi \{ Z_1 \in A \} = \pi(A) \)
- Implies stationarity when initial dist’n is \( \pi \)

When is \( \{ (S_n, C_n) : n \geq 0 \} \) (positive) Harris recurrent?
Some Stability Conditions

- **Density component** $g$ of a cdf $F$: $F(t) \geq \int_0^t g(u) \, du$

- $s \rightarrow s'$ iff $p(s'; s, e) > 0$ for some $e$

- $s \sim s'$: either $s \rightarrow s'$ or $s \rightarrow s^{(1)} \rightarrow \cdots \rightarrow s^{(n)} \rightarrow s'$

- **Assumption PD($q$):**
  - State space $S$ is finite
  - GSMP is irreducible: $s \sim s'$ for all $s, s' \in S$
  - There exists $\bar{x} \in (0, \infty)$ s.t. all clock-setting dist’n functions
    - Have finite $q$th moment
    - Have density component positive on $[0, \bar{x}]$
Positive Harris Recurrence in Light-Tailed GSMPs [Haas99]

- $\bar{\phi} (\{s\} \times A) = \text{Lebesgue measure of } A \cap [0, \bar{x}]^M$
- Theorem: If Assumption PD(1) holds, then the $(S_n, C_n)$ chain is positive Harris recurrent with recurrence measure $\bar{\phi}$
- Implies $P_{\mu} \{ S_n = s \text{ i.o.} \} = 1$ for all $s \in S$ with finite expected (continuous-time) hitting times
- Proof:
  - Show that chain is “$\bar{\phi}$-irreducible”
  - Establish Lyapunov drift condition and apply MC machinery (Meyn and Tweedie, 1993)
- Alternate approach to recurrence: geometric-trials arguments
  - Can drop positive-density assumption
  - Use detailed analysis of specific GSMP structure
Positive Recurrence in Heavy-Tailed GSMPs: It Depends

- Example 1: Uninfluential events
  - For all $s \in S$: $e \in E(s)$ and $p(s; s, e) = 1$
  - No effect on state or other clocks
  - If all heavy-tailed events are uninfluential and PD(1) holds otherwise, then positive recurrence

- Example 2: $e_1$ is heavy-tailed, so no positive-recurrence
Recurrence in Heavy-Tailed GSMPs

- $S_n = \text{state just after } n\text{th state transition}$
- Conjecture: $P\{ S_n = s \text{ i.o.} \} = 1 \text{ for each } s \text{ under PD}(0)$
  - State space $S$ is finite
  - GSMP is irreducible
  - $\exists \bar{x} > 0 \text{ s.t. each } F(\cdot; e) \text{ has positive density on } (0, \bar{x})$
- Certainly true for CTMCs
- CONJECTURE IS FALSE for GSMPs!
  - In the presence of heavy-tailed clock-setting dist’ns
A Counterexample

- $S = \{ 1, 2, 3 \}$ and $E = \{ e_1, e_2, e_3 \}$
- Event sets: $E(s) = \{ e_1, e_2, e_3 \}$ for all $s$ (renewal processes)
- $p(s'; s, e^*) = 0$ or $1$
- Clock-setting distributions:
  - $F(t; e_1) = 1 - (1 + t)^{-\alpha}$
  - $F(t; e_2) = 1 - (1 + t)^{-\beta}$
  - $F(\cdot; e_3)$ is Uniform$[0, a]$ with $\beta > 1/2$ and $\alpha + \beta < 1$
- GSMP hits state $s = 2$ only if:
  - $e_1$ occurs and then $e_2$ occurs with no intervening occurrence of $e_3$
- Claim: $P\{ S_n = 2 \text{ i.o.} \} = 0$
- Intuition: heavy clocks are rarely small simultaneously
Proof

- Observe: \( P \{ S_n = 2 \text{ i.o.} \} = 0 \) iff \( P \{ B_n \text{ i.o.} \} = 0 \)
  - \( B_n = \{ C_2(T_n) \leq C_3(T_n) \} \)
  - \( T_n = \text{nth occurrence time of } e_1 \)
  - \( C_i(t) = \text{clock reading for } e_i \text{ at time } t \)

- Borel-Cantelli: suffices to show that \( \sum_{n=1}^{\infty} P \{ B_n \} < \infty \)

- Bound \( P \{ B_n \} \) by an integral:
  \[
P \{ B_n \} \leq \int_0^{\infty} P \{ C_2(t) \leq a \} f_1^*(t) \, dt
  \]
  - \( f_1^* = \text{density of } T_n = \text{n-fold convolution of } f(\cdot; e_1) \)

- Sum over \( n \):
  \[
  \sum_{n=1}^{\infty} P \{ B_n \} \leq \int_0^{\infty} h(t)u_1(t) \, dt
  \]
  - \( h(t) = P \{ C_2(t) \leq a \} \)
  - \( u_1 = \text{renewal density function for } F(\cdot; e_1) \)

- So suffices to show that \( \int_0^{\infty} h(t)u_1(t) \, dt < \infty \)
Proof–Continued

- To show: \( \int_{0}^{\infty} h(t)u_1(t) \, dt < \infty \) where \( h(t) = P\{ C_2(t) \leq a \} \)
- Renewal argument: \( h = U_2 \ast Q \)
  - \( U_2 \) = renewal function for \( F(\cdot; e_2) \)
  - \( Q(t) = F(t + a; e_2) - F(t; e_2) \)
- Heavy-tail key renewal theorem [Erickson70]:
  - Light-tail KRT: \( h(t) \to m_2(a)/m_2(\infty) \)
  - where \( m_2(t) = \int_{0}^{t} \bar{F}(u; e_2) \, du = (1 + t)^{1-\beta}/(1 - \beta) \)
  - \( \bar{F} = 1 - F \)
- Regular variation: \( G \in RV_\lambda \) iff \( \lim_{t \to \infty} G(tx)/G(t) = x^\lambda \)
- Erickson: \( h(t) = O(1/m_2(t)) \)
- Thus \( h(t) = O(t^{\beta-1}) \)

\[
\begin{align*}
N(t) & \quad \text{X_i} \sim F \\
& \quad 0 \quad 1 \quad 2 \quad 3 \\
& \quad X_1 \quad T_1 \quad X_2 \quad T_2 \quad X_3 \quad T_3 \\
& \quad C(t) \quad \text{Excess life} \\
U(t) & = E[N(t)] = \sum_{n=1}^{\infty} F^n(t) \\
u(t) & = dU(t)/dt = \sum_{n=1}^{\infty} f^n(t)
\end{align*}
\]
Proof–Continued

- To show: \( \int_0^\infty h(t)u_1(t) \, dt < \infty \)
  where \( h(t) = O(t^{\beta - 1}) \)

- Tauberian theorems: \( \bar{F}(\cdot; e_1) \in RV_{-\alpha} \Rightarrow U_1 \in RV_{\alpha} \)
  - \( U_1 = \text{renewal function for } F(\cdot; e_1) \)

- Landau’s theorem: \( u_1 \) ultimately monotone \( \Rightarrow u_1 \in RV_{\alpha - 1} \)
  - Temporarily assume ultimate monotonicity
  - Result in Feller \( \Rightarrow u_1(t) = O(t^{\alpha - 1 + \epsilon}) \)

- Combine: \( \int_0^\infty h(t)u_1(t) \, dt = \int_0^\infty O(t^{\alpha + \beta + \epsilon - 2}) \, dt < \infty \)
  since \( \alpha + \beta + \epsilon < 1 \)
Ultimate-Monotonicity Proof (Sketch)

- \( \mathcal{L}(f_1)(s) = \alpha s^\alpha e^s \Gamma(-\alpha, s) \), where
  \[ \Gamma(a, s) = \int_s^\infty e^{-t} t^{a-1} \, dt \]

- \( \mathcal{L}(u_1)(s) = \frac{\mathcal{L}(f_1)(s)}{1 - \mathcal{L}(f_1)(s)} \)

- \( \mathcal{L}(u'_1)(s) = \frac{s \mathcal{L}(u_1)(s)}{1 - \mathcal{L}(u_1)(s)} - u_1(0+) = g(s) \), where
  \[ g(s) = \frac{\alpha s^{\alpha+1} e^s \Gamma(-\alpha, s)}{1 - \alpha s^\alpha e^s \Gamma(-\alpha, s)} - \alpha \]

- \( g(s) \) is analytic except at origin
Ultimate-Monotonicity Proof — Continued

- Inversion formula: \( u_1'(t) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^{st} g(s) \, ds \)

- Apply Cauchy’s integral theorem:

\[
u_1'(t) \approx -\frac{ct^{\alpha-2} \sin(\pi(1-\alpha))}{\pi} < 0\]
### Summary and Conjecture

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<th># HT Clocks</th>
<th>RW Equiv.</th>
<th>Recurrent?</th>
<th>&quot;Positive Recurrent&quot;?</th>
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<td>Yes</td>
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<tr>
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<td>2</td>
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<td>Depends</td>
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<tr>
<td>≥ 2</td>
<td>≥ 4</td>
<td>Depends</td>
<td>Depends</td>
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</table>
A Special Case

- Hazard rate: $h(x; e) = f(x; e)/\bar{F}(x; e)$
- Theorem: $P\{ S_n = s \text{ i.o.}\} = 1$ for each $s$ if
  - State space $S$ is finite
  - GSMP is irreducible
  - $\exists \bar{x} > 0$ s.t. each $F(\cdot; e)$ has positive density on $(0, \bar{x})$
  - At most one active event with heavy-tailed clock-setting dist’n
  - $\alpha(e) \leq h(x; e) \leq \beta(e)$ for each light-tailed event $e$
- Proof uses regenerative structure [Glynn89] + geometric trials
Questions?

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