On Non-crossing (Projected) Spanning Trees of 3D point sets

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1 Introduction

We study the problem of optimizing the spanning tree of a set of points in $\mathbb{R}_3$, whose projection onto a given plane contains no crossing edges. Denoted by NCMST (which stands for non-crossing minimum spanning trees), this problem is defined as follows. Given a set of points $P$ in $\mathbb{R}_3$ and a plane $F$, find a spanning tree whose projection onto $F$ contains no crossing edges and its length is minimum among all such spanning trees. MCMST is motivated by areas such as shape modeling, surface reconstruction, and more. We prove that this problem is NP-complete and show that greedy algorithms (analogous to Prim and Kruskal), may perform arbitrarily bad. Nevertheless, we report that experimentally these algorithms perform well in practice.

Figure 1: A reduction from $\sigma = x(x + y)(\bar{x} + \bar{y})\bar{y}$. Negated literals are directed to the clauses below. Here $\sigma$ is satisfied by assigning true and false to $x$ and $y$ respectively.

2 Proof of NP-Completeness

In order to prove hardness of NCMST, we use a reduction from positive-negative planar 3-SAT (PN-planar 3-SAT for short). It is a special restriction of the planar 3-SAT which remains NP-complete [1] and whose restriction is defined as follows. Let $\sigma$ be a formula in CNF form and let $G_\sigma = (V, E)$ be the corresponding graph of $\sigma$. $V = X \cup C$ includes the variables $x_i$ and the clauses $C$, of $\sigma$ and $E = E_1 \cup E_2$ where $E_1 = \{(x_i, x_{i+1}) | 1 \leq i < n\} \cup \{(x_n, x_1)\}$ ($n$ is the number of variables in $\sigma$) and $E_2 = \{(c_i, x_j) | clause c_i contains variable x_j\}$. In this embedding, the edges of $E_1$ partition the plane into two faces, $F_1$ and $F_2$. An instance of an PN-planar 3-SAT satisfies the following restriction. If two clauses $c_1$ and $c_2$ contain opposite occurrences of some variable $x$, then one lies inside $F_1$ and the other inside $F_2$. We denote the decision problem of NCMST by DNCMST.

Theorem 2.1 DCMST is NP-complete.

Proof: The problem is in NP: Given a spanning tree $S$, maximum length $K$ and a plane $F$, we can verify that the sum of the lengths of the edges of $S$ is at most $K$ and that the corresponding projection has no crossing edges. All can be carried out easily in polynomial time.

We next describe a polynomial reduction from PN-planar 3-SAT. An illustration of the reduction is given in Figure 1 where different points are drawn with different shapes and textures for clarity. The figure illustrates the projections of the points onto the $xy$-plane (where $z = 0$). The orientation of the $xy$-plane follows the standards (the $x$ axis spans from left to right and the $y$ axis from bottom to top). Each variable $v_i$ of $\sigma$ is assigned a gadget which consists of points along three edges of a rectangle $R(v_i)$ (left, bottom and top, denoted by $l(v_i)$, $b(v_i)$ and $t(v_i)$ respectively) and another special point denoted by $s(v_i)$. In the figure which corresponds to a formula of two variables, $x$ and $y$, the white small discs are the points of $R(x)$ and $R(y)$ and the two squares represent $s(x)$ and $s(y)$. For any variable $v_i$, $s(v_i)$ is located on the middle of $r(v_i)$, the imaginary right edge of $R(v_i)$. All these gadgets are identical, aligned horizontally and long enough to contain all the wires that we describe next. The $n$ variables gadgets are connected by $n - 1$ couples of wires. One of the wires of any couple passes slightly above $s(v_i)$ and the other passes slightly below. In the example, those are the two shaded wires. These wires also partition the $xy$-plane into two parts which are analogous to the partition in the PN-planar 3-SAT problem. The points represented by big white discs are the gadgets for the clauses. Wires connecting variable gadgets and clause gadgets correspond to inclusions of literals in clauses of $\sigma$. These wires are connected in both sides to $\{l(v_i) | 1 \leq i \leq n\}$ from inside the variable gadgets and to the clause points. Since $\sigma$ satisfies embedding of PN-planar 3-SAT, opposite literals will go to different directions (up and down) in our reduction. We assign the $z$ values of the points as follows. All the points in our reduction except from the points $\cup_{1 \leq i \leq n} \{b(v_i) \cup t(v_i) \cup s(v_i)\}$ have $z$ value $M$ where $M$ is much bigger than the diameter length any variable gadget. $b(v_i)$ and $t(v_i)$ slide down linearly from left, where the $z$ value is $M$, all the way to the right, where the $z$ value is $0$ (both have the same structure). The $z$ value of $s(v_i)$ is $0$ as well. It is known that drawing planar graphs with grid points requires polynomial number of grid points. Thus,

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1For simplification purposes, the corresponding formula clauses contain only one and two literals.
2By 'above' and 'below' we refer to larger and smaller $y$ values on the projection respectively.
our reduction generates a set of \( \mathcal{P}(\sigma) \) points where \( \mathcal{P} \) is some polynomial function of \( \sigma \). In our reduction, the distance between any two adjacent points is a constant denoted by \( \varepsilon \). We make sure that any two wires connecting two variable gadgets or variable gadgets to clause gadgets are sufficiently separated, meaning that their separation is much bigger than \( \varepsilon \). Let the distance between \( b(v_i) \) and \( s(v_i) \) be denoted by \( \delta \) (this is also the distance between \( t(v_i) \) and \( s(v_i) \)). It is fixed in all variable gadgets. We make sure that \( \delta >> \varepsilon \). For some formula \( \sigma \), let \( K = \mathcal{P}(\sigma) \). Note that \( n \) points in the reduction are denoted by \( S(\sigma) = \{s(v_i) | 1 \leq i \leq n \} \). Under this construction, any spanning tree will have \( K - 1 \) links.

We next prove that the size of the MST in our reduction (denoted by \( L(\sigma) \)) is at most \((K - n - 1)\varepsilon + n\delta\) if and only if \( \sigma \) is satisfiable. Otherwise \( L(\sigma) \) is bigger. We first note (based on simple observations on our construction) that the length of any spanning tree is never smaller than \( (K - n - 1)\varepsilon + n\delta \). Thus, we exclude such cases. Suppose \( \sigma \) is satisfiable. We first connect all the points of \( S(\sigma) \) to either \( b(v_i) \) or \( t(v_i) \), depending on the value of \( v_i \) in \( \sigma \). This connection will be such that the wires that will correspond to true values of \( v_i \) will not be cut by this connection. We refer to these connections by gates. Now, the wires that correspond to truth values can be connected to their corresponding clauses without crossing any edge on the projected \( xy \)-plane. Since \( \sigma \) is satisfiable, all clauses will be connected in this way. All wires that are cut by any of the gates, can be connected in both sides from the variable gadgets’ left walls and from one of the clause gadgets. As for the variable connecting wires (the shaded wires in the figure), one of them will be able to connect its two variable gadgets and the other will be connected by both gadgets from left and right. Based on this construction, all points are connected by a spanning tree with a length \( L(\sigma) = (K - n - 1)\varepsilon + n\delta \). On the other hand, suppose that an MST has length of \((K - n - 1)\varepsilon + n\delta \). Based on our construction, it must be the case that all the points except the points of \( S(\sigma) \) are connected, with links of length \( \varepsilon \) and the points of \( S(V) \) are connected to exactly one point above or below (in the \( xy \) projected plane) which belongs to variable gadgets, with links of length \( \delta \). This connection determines the truth value of the variables as explained above and these truth values satisfy all the clauses.

\[ \Box \]

3 Performance of Greedy Heuristics

A natural heuristic for obtaining good results for NCMST would be to use greedy heuristics, analogous to Kruskal or Prim, with the constraint that the projection of the next edge \( e \) does not intersect the projection of the current tree. We call the corresponding MST algorithms by either constrained-Prim or constrained-Kruskal. We next show that such greedy heuristics can produce arbitrarily bad results. Consider the points in the left side of Figure 2. It consist of two ‘stars’

![Figure 2: Left: The spanning tree obtained with the constrained-Prim algorithm. Thick edges connecting white to black points. Right: Connecting the points using two spiral sequences of links requires only one edge that connects white and black points (in this case it connects the center points, thus invisible here).](image)

References


\[ \text{The adjacency relation here is implied from our description as points that are consecutive in wire chains, variable gadgets, etc.} \]